B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6DSE33

(Group Theory II)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

Group-A

1. Answer *any ten* questions: 2×10=20

(a) Show that the set of automorphisms of a group forms a group under the operation of composition of functions.

(b) Let $f : G \to G$ be a mapping defined by $f(x) = x^{-1}, \forall x \in G$. Show that f is an automorphism if G is abelian.

(c) Let G be a group and $g \in G$. Show that the mapping $\phi_g(x) = gxg^{-1}$ for all $x \in G$ is an automorphism.

(d) Prove that a commutative group of order 10 is cyclic.

(e) Show that the direct product $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group.

(f) Show that the direct product $S_3 \times \mathbb{Z}$ of the groups S_3 and $\mathbb{Z}$ is an infinite non-commutative group.

(g) Let G_1, G_2 be two commutative groups. Show that direct product $G_1 \times G_2$ is a commutative group.

(h) State fundamental theorem for finite abelian groups.

(i) Find all sylow 2-subgroups of A_4.

(j) Show that any group of order p^2 is commutative, where p is a prime.

(k) Prove that no group of order 8 is simple.

(l) State Sylow's first theorem.

(m) Prove that a group of order 99 has a unique normal subgroup of order 11.

(n) Prove that a group G is commutative if and only if $G' = \{e\}$.

(o) Write the class equation of S_3.

Group-B

2. Answer *any four* questions: 5×4=20

(a) If G is an infinite cyclic group, then prove that $Aut(G)$ is a group of order 2.

(b) Show that every characteristic subgroup of a group G is a normal subgroup of G. Is the converse true? Support your answer. 3+2

(c) Show that the derived subgroup G' of a group G is a normal subgroup of G and $G/_{G'}$ is commutative. 3+2

(d) State and prove Cauchy's theorem for finite group. 1+4

(e) Prove that any group of order 30 is not simple.

(f) Show that every group of order 255 is cyclic.

Group-C

3. Answer *any two* questions: 10×2=20

(a) (i) Let G be a group and $Z(G)$ be the centre of the group G. Then show that $Inn(G)$ is isomorphic to the quotient group $G/_{Z(G)}$.

(ii) Find the number of inner automorphisms of the group S_3.

(iii) Show that $Aut(\mathbb{Z}_n) \simeq U_n$ 4+3+3

(b) (i) Let H and K be two finite cyclic groups of order m and n respectively. Prove that the direct product $H \times K$ is a cyclic group if and only if $gcd(m, n) = 1$.

(ii) Find the number of elements of order 5 in $\mathbb{Z}_{15} \times \mathbb{Z}_5$.

(iii) Let G be a finite p-group with $|G| > 1$. Prove that $|Z(G)| > 1$. 4+3+3

(c) (i) Let p be an odd prime. If G is a group of order $2p$, then show that either $G \cong \mathbb{Z}_{2p}$ or $G \cong D_p$.

(ii) Show that every group of order 99 is abelian.

(iii) Show that every cyclic group is abelian. 4+3+3

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